Indian Statistical Institute, Bangalore Centre
B.Math. (III Year)/ M.Math. (II year) : 2014-2015
Semester I : Semestral Examination
Markov Chains

14.11.2014 Time: 3 hours Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. (8 + 12 = 20 marks) Let $\{X_n : n \geq 0\}$ be a Markov chain on $\{0,1\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

Define $Z_n = (X_{n-1}, X_n)$ for $n = 1, 2, 3, \ldots$

(i) Show that $\{Z_n : n \geq 1\}$ is a Markov chain whose state space is the four point set $\{(0,0),(0,1),(1,0),(1,1)\}$.

(ii) Determine the transition probability matrix of $\{Z_n\}$.

2. (25 marks) $\{X_n : n = 0,1,2,\cdots\}, \{Y_n : n = 0,1,2,\cdots\}$ are independent, irreducible, aperiodic, positive recurrent Markov chains on a countable state space S with the same transition probability matrix $P = ((P_{ij}))$. Let $T = \min\{n \geq 1 : X_n = Y_n\}$. Show that $T < \infty$ with probability one. (Hint: Consider $\{(X_n, Y_n) : n \geq 0\}$.)

3. (7 + 13 = 20 marks) Let $0 < p < 1$, and $\{X_n : n = 0,1,2,\cdots\}$ be a Markov chain on $S = \{1,2,3\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 0 & (1-p) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(i) Find the period of each state.

(ii) What can you say about $\lim_{n\to\infty} \mathbf{P}^n$?

4. (20 marks) $\{N(t) : t \geq 0\}, \{M(s) : s \geq 0\}$ are independent Poisson processes with respective arrival rates $\lambda, \mu > 0$. Let $T = \inf\{s \geq 0 : M(s) = 1\}$. Find the probability mass function of $N(T)$.

5. (20 marks) Suppose that shocks to a system occur according to a time homogeneous Poisson process $N(\cdot)$ with arrival rate $\lambda > 0$; let Y_k denote the $k-$th shock. Assume that $\{N(t) : t \geq 0\}$ and $\{Y_i : i = 1, 2, \cdots\}$ are independent families of random variables, and that $\{Y_i : i \geq 1\}$ is a sequence of i.i.d. positive random variables with mean $\mu > 0$. Assume also that the amplitude of a shock decreases with time at an exponential rate α. Let $X(t)$ denote the sum of all amplitudes by time t. Find $E[X(t)],\ t > 0$.